Riemannian Geometry | Week 10

Last week, we saw the definition of the Riemann curvature tensor:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and $$\hat{R}(X,Y,Z,W) = \langle R(X,Y)W, Z \rangle$$

In components: $R^{a}{}_{\beta\gamma\delta} = (R(\partial_\gamma, \partial_\delta)\partial_\beta)^a$

$$R_{a\beta\gamma\delta} = g_{a\lambda} R^{\lambda}{}_{\beta\gamma\delta}$$

and $\hat{R}_{a\beta\gamma\delta} = R_{a\beta\gamma\delta}$, $\quad \hat{R}_{a\beta\gamma\delta} = \hat{R}(\partial_a, \partial_\beta, \partial_\gamma, \partial_\delta)$

$$(= \langle R(\partial_a, \partial_\beta)\partial_\delta, \partial_\gamma \rangle$$

Symmetries:

$R_{a\beta\gamma\delta} = -R_{\beta a\gamma\delta} = -R_{a\beta\delta\gamma}$

$R_{a\beta\gamma\delta} = R_{\gamma\delta a\beta}$

1^{st} Bianchi, 2^{nd} Bianchi

Sectional curvature: $$K(\Pi) = \frac{R(X,Y,X,Y)}{\|X\wedge Y\|^2}$$

"Π = spanned by X, Y

(If e_1, e_2 orthonormal, $X = X^1 e_1 + X^2 e_2$, $Y = Y^1 e_1 + Y^2 e_2$:

$$\|X \wedge Y\| = \left|\det\begin{pmatrix} X^1 & X^2 \\ Y^1 & Y^2 \end{pmatrix}\right| \quad)$$

Ricci tensor: $$Ric(X,Y) = \mathrm{Trace}\,[Z \longrightarrow R(Z,X)Y]$$

$$= \sum_{i=1}^{n} (R(\partial_i, X)Y)^i$$

$$= R^{i}{}_{jik} X^j Y^k$$

So $Ric_{jk} = R^{i}{}_{jik} = g^{a\beta} R_{a j\beta k} \overset{\text{symmetries}}{=} g^{a\beta} R_{ja k\beta}$

- $Ric(V,W) = Ric(W,V)$ (see the exercise)

In an orthonormal basis e_i:

$$Ric(V,W) = \sum_{i=1}^{n} R(e_i, V, e_i, W)$$

- Scalar curvature:

$$S = g^{\alpha\beta} Ric_{\alpha\beta}$$

$$\overset{\text{orthonormal}}{\Longrightarrow} S = \sum_{i=1}^{n} Ric(e_i, e_i) = \sum_{i,j=1}^{n} R(e_i, e_j, e_i, e_j)$$

On a space of constant sect. curvature K:

$$R(e_i, e_j, e_i, e_j) = \begin{cases} K, & i \neq j \\ 0, & i = j \end{cases}$$

$$\Rightarrow S = n(n-1)K$$

$$Ric(e_j, e_j) = (n-1)K \xrightarrow{e_j \text{ can be any vector in } T_pM} Ric = (n-1) \cdot K \cdot g$$

Submanifolds and 2^{nd} fundamental form

Let (M^m, g) be a Riem. manifold and $N^n \subset M^m$ a submanifold.

$\forall p \in N$: $\underbrace{T_pN}_{\dim = n} \subseteq T_pM$: Tangent vectors at N

$$\underbrace{T_pN^\perp}_{\dim = m-n} = \{ v \in T_pM : v \perp T_pN \}$$

Let $\Pi_p^T : T_pM \to T_pN$, $\Pi_p^\perp : T_pM \to T_pN^\perp$ be the orthogonal projections

1^{st} fundamental form: The induced metric

$$\bar{g}_p = g_p|_{T_pN}$$

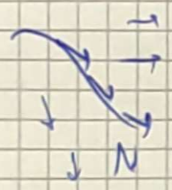
Define: $\forall X, Y \in \Gamma(M)$ • $\bar{\nabla}_X Y := (\nabla_X Y)^T \quad (= \pi^T(\nabla_X Y))$

• $B(X,Y) = (\nabla_X Y)^{\perp}$

So $\underset{}{\nabla_X Y} = \underset{\in TN}{\bar{\nabla}_X Y} + \underset{\in TN^{\perp}}{B(X,Y)}$

Set of vector fields on M which are tangent to N:

$$\Gamma(M,N) = \{ X \in \Gamma(M) \mid X_p \in T_pN \quad \forall p \in N \}$$



Note: Any tangent vector field to N can be extended as a vector field in $\Gamma(M,N)$

Lemma:

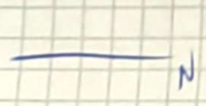
a) Let $X \in \Gamma(M)$. $X \in \Gamma(M,N)$ iff $X(f)|_p = 0$ $\forall p \in N$, $\forall f$ constant on N.

b) If $X, Y \in \Gamma(M,N) \to [X,Y] \in \Gamma(M,N)$

c) $\Gamma(M,N) \to \Gamma(N)$ is onto, i.e. any $X \in \Gamma(N)$ admits an extension in $\Gamma(M,N)$

Sketch of proof: Locally, around any $p \in N$, we can find coordinates $(x^1, \dots, x^m)$ such that $N = \{x^{n+1} = \dots = x^m = 0\}$

Then $X = X^i \partial_i \in \Gamma(M,N)$ if $X^i|_N = 0$ for $i = n+1, \dots, m$.



Proposition

$\bar{\nabla}$ is the Levi-Civita connection of $\bar{g}$

Proof: We need to verify the 2 properties:

1) $\bar{\nabla}$ is torsion free: If $X, Y \in \Gamma(M,N)$ (extended from $\Gamma(N)$),

$$\bar{\nabla}_X Y - \bar{\nabla}_Y X = (\nabla_X Y)^T - (\nabla_Y X)^T = (\nabla_X Y - \nabla_Y X)^T = ([X,Y])^T = [X,Y]$$

since $[X,Y] \in \Gamma(M,N)$

2) $\bar{\nabla}$ respects $\bar{g}$:

If $X, Y, Z \in \Gamma(M,N)$

$$X(\bar{g}(Y,Z)) = X(g(Y,Z)) = g(\nabla_X Y, Z) + g(\cancel{\ }Y, \nabla_X Z)$$

$$= g(\bar{\nabla}_X Y + \underbrace{B(X,Y)}_{\text{orthogonal to } TN!}, Z) + \dots$$

$$= g(\bar{\nabla}_X Y, Z) + g(\dots$$

$$= \bar{g}(\bar{\nabla}_X Y, Z) + \dots$$

$\square$

~~The restriction of B on N (i.e. applied~~ to

B is the second fundamental form:

Prop: The 2nd fund form of a submanifold $N \subset (M,g)$ has the following properties:

a) B is $C^\infty(N)$-bilinear (i.e. it is a $(0,2)$-tensor)

b) $\forall p \in N$: $B|_p : T_pN \times T_pN \longrightarrow T_pN^\perp$ is well defined

c) B is symmetric

<u>Proof</u>: $B(X,Y) = (\nabla_X Y)^{\perp} \quad \forall\ X,Y \in \Gamma(M,N)$

a) B is obviously $\mathbb{R}$-linear, and if $f \in C^{\infty}(N)$: Extend (arbitrarily) to $\tilde{f} \in C^{\infty}$

$$B(fX, Y) = f\,B(X,Y) \quad (\text{by } C^{\infty}\text{-linearity of } \nabla \text{ in } 1^{st} \text{ argument})$$

$$B(X, fY) = (\nabla_X (fY))^{\perp} = (f \cdot \nabla_X Y + X(f) \cdot Y)^{\perp} =$$

$$= f\,(\nabla_X Y)^{\perp} + X(f) \cdot \cancel{Y^{\perp}}^{\,0}$$

$$= f \cdot B(X,Y)$$

b) is a consequence of a)

c) $B(X,Y) - B(Y,X) = (\nabla_X Y - \nabla_Y X)^{\perp} = ([X,Y])^{\perp} = 0$

($[X,Y] \in \Gamma(M,N)$ if $X,Y \in \Gamma(M,N)$)

As in the case of surfaces in $\mathbb{R}^3$: The 2^{nd} fundamental form measures the acceleration (in M) of geodesics in N:

<u>Lemma</u>: If γ is a geodesic of $(N, \bar{g})$, then

$$B(\dot\gamma, \dot\gamma) = \nabla_{\dot\gamma} \dot\gamma$$

<u>Proof</u>: $\bar\nabla_{\dot\gamma} \dot\gamma = 0$

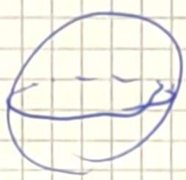
So $\nabla_{\dot\gamma} \dot\gamma = \bar\nabla_{\dot\gamma} \dot\gamma + B(\dot\gamma, \dot\gamma)$ ∎

<u>Def</u>: $N \subseteq (M,g)$ is totally geodesic if every geodesic in N is a geodesic in M

$\iff B \equiv 0$

Example: - A plane in $\mathbb{R}^n$

- The equator of S^n
  $\cong S^{n-1}$



The fundamental equation for a Riem. submanifold

Prop: (The Weingarten equation)

Let $N \subset (M,g)$ be a submanifold and $X, Y \in \Gamma(M, N)$,

$W \in \Gamma(M)$ with $W \perp TN$ along N

Then at every point on N:

$$g(\nabla_X W, Y) = -g(W, B(X,Y))$$

Proof: Along N: $g(W,Y) = 0$, so $(X \in \Gamma(M,N))$:

$$0 = X(g(W,Y)) = g(\nabla_X W, Y) + g(W, \nabla_X Y)$$

$$= g(\nabla_X W, Y) + g(W, \bar{\nabla}_X Y + B(X,Y))$$

orthogonal

$\blacksquare$

Using the 2nd-fundamental form, we can obtain information about the intrinsic geometry of N:

Theorem (Gauss equation): $\forall X, Y, Z, W \in T_pN$:

$$\bar{R}(X,Y,Z,W) - R(X,Y,Z,W) = \langle B(X,Z), B(Y,W)\rangle - \langle B(X,W), B(Y,Z)\rangle$$

Riem. curv. tensor of $\bar{g}$ ... of g

Before we go to the proof, let's see some corollaries:

<u>Cor</u>: If $N \subset (M,g)$ is totally geodesic:

$$\overline{R}(X,Y)Z = R(X,Y)Z \quad \forall\ X, Y, Z \in \Gamma(N).$$

<u>Cor</u>: If $\Pi \subseteq T_pN$ is a 2-plane:

$$\overline{K}(\Pi) = K(\Pi) + \frac{\langle B(X,X), B(Y,Y)\rangle - \|B(X,Y)\|^2}{\|X \wedge Y\|^2}$$

(Useful for surfaces in $\mathbb{R}^3$: $\overline{K}$ is the Gauss-curvature)

<u>Hypersurfaces</u>:

$N \subset M$ is a hypersurface if $\dim N = \dim M - 1$

Then: $\forall p \in N$: $(T_pN)^{\perp}$ has dimension 1.

<u>Def</u>: A <u>co-orientation</u> of N: a choice of smooth unit vector $\hat{n}$ orthogonal to N

(locally: every hypersurface has a co-orientation)

Scalar 2nd fundamental form:

(0,2) tensor b,

$$b(X,Y) = g(B(X,Y), \hat{n}) \quad \Leftrightarrow \quad B(X,Y) = b(X,Y)\cdot \hat{n}$$

$$= g(\nabla_X Y, \hat{n})$$

$\overset{\text{Weingarten}}{\Longrightarrow}$ $b(X,Y) = -g(\nabla_X \hat{n}, Y)$.